OPTIMAL ADVERTISING POLICY UNDER DYNAMIC CONDITIONS

BY
MARC NERLOVE AND KENNETH J. ARROW

TECHNICAL REPORT NO. 102
DECEMBER 11, 1961

PREPARED UNDER CONTRACT Nonr-225(50)
(NR-047-004)
FOR
OFFICE OF NAVAL RESEARCH

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
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Marc Nerlove and Kenneth J. Arrow
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Advertising expenditures are similar in many ways to investments in durable plant and equipment. The latter affect the present and future character of output and, hence, the present and future net revenue of the investing firm. Advertising expenditures affect the present and future demand for the product and, hence, the present and future net revenue of the firm which advertises.[1] In a previous paper, Dorfman and Steiner [4] have given the necessary conditions for maximum net revenue when: (a) price and advertising expenditures are the only variables affecting the demand for the product; (b) current advertising expenditures do not affect the future demand for the product, and (c) the decision-maker is a monopolist who can determine both price and advertising expenditures. They have also extended their analysis to cover the case in which the quality of the product is variable.

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* The research on which this paper is based was supported in part by a grant from the National Science Foundation to Stanford University and in part by Contract Nonr-225(50) between the Office of Naval Research and Stanford University. An earlier draft with the same title by Nerlove was supported by the Department of Agricultural Economics at the University of Minnesota.

We are indebted to O. Brownlee, I. R. Savage, R. Strotz, F. V. Waugh, and S. Weintraub for helpful comments on an earlier draft.

In what follows, the Dorfman-Steiner model is extended to cover the situation in which present advertising expenditures affect the future demand for the product. It is shown that, under plausible assumptions, the necessary conditions for a maximum of the present value of future net revenues lead to a decision rule which is similar to that actually used by many firms. The Dorfman-Steiner model is a special case of the model presented here.

1. Formulation of the Model and the Optimal Price Policy

The demand for the output of an individual firm or of an industry depends on advertising expenditures in addition to the price of the product, consumer incomes, and the prices of competing or complementary products. Advertising expenditures may shift the demand function by adding new customers, those who may never have consumed the product before in the case of an industry, or those who have previously consumed the product of another firm in the case of an individual firm. Such expenditures may also alter the tastes and preferences of consumers and thereby change the shape of the demand function as well as shift it. For example, "brand" advertising may make the price elasticity of demand for the brand advertised lower than it would otherwise be. On the other hand, the attraction of new customers by means of advertising and the consequent broadening of the market may make demand more sensitive to price.[2]

Regardless of its precise effects on the demand function, advertising expenditure at any one time may be expected to lose its effectiveness in subsequent periods. An advertising campaign conducted now may bring a hundred thousand customers into the fold today, but next month or next year many of these will have drifted off. Other firms and other industries do not stand still but also commit funds to advertising; these campaigns in turn draw customers to the products or brands advertised and away from the product or brand initially considered. Furthermore, permanent changes in consumer tastes and preferences are difficult to effect; while a strenuous advertising campaign may induce a change in tastes and preferences for a time, there is a tendency for the preferences of consumers to return to their old pattern. On the other hand, the effects of a given advertising campaign, both upon the number of consumers and their tastes, tend to persist for a considerable period following the campaign, albeit, for the reasons given, to a steadily diminishing extent.[3]

One possibility of representing the temporal differences in the effects of advertising on demand would be to include a number of dated, past advertising outlays in the demand function. However, such an approach is not especially useful. A more promising analytical approach, and one which has considerable intuitive appeal, is to define a stock, which we shall call <u>good</u> <u>will</u> and denote by $A(t)$, and which we suppose summarizes the effects of current and past advertising outlays on demand. The price of a unit of good will, we shall suppose, is $1, so that a dollar of current advertising expenditure increases good will by a like amount.[4] On the other hand, a dollar spent some time ago should, according to our previous argument, contribute less. One possible way of representing this lesser contribution is to say that good will, like many other capital goods, depreciates. If we further assume that current advertising expenditure cannot be negative[5] and that depreciation occurs at a constant proportional rate, δ , we have

(1) $$\dot{A} + \delta A = a \geq 0 \quad ,$$

where a is current advertising outlay, a and A are understood to be functions of time, and the dot denotes differentiation with respect to time. Equation (1) states that the <u>net</u> investment in good will is the difference between the <u>gross</u> investment (current advertising outlay) and the depreciation of the stock of good will.[6]

We are now in a position to formulate our model: Let $q(t)$ be the rate at which purchases are made at time t , $p(t)$ the price charged, and $z(t)$ other variables not under the control of the firm such as consumer incomes, population, and the prices of substitute and complementary products. The quantity demanded is assumed to depend on $p(t)$, $A(t)$, and $z(t)$:

(2) $$q = f(p,A,z) \quad .$$

The rate at which total production costs, $c(t)$, are incurred is assumed to be a function of output:

(3) $$c(t) = C(q) \quad .$$

Let $r(t)$ be the rate at which revenue net of production costs and current advertising outlays accrues to the firm; then

(4) $$\begin{aligned} r(t) &= pf(p,A,z) - C(q) - a \\ &= R(p,A,z) - a \quad , \end{aligned}$$

where R is revenue net of production expenses only. We assume that the firm attempts to maximize the present value of the stream of revenues net of both production expenses and advertising costs by appropriate price and advertising policies over time. That is, for a given initial value of A^* ,

$$A(0) = A_o \quad ,$$

the time paths of p and A are chosen to maximize

$$(5) \qquad V\left\{p,A\right\} = \int_0^{\infty} e^{-\alpha t}[R(p,A,z) - a]dt \quad ,$$

subject to (1), where α is a fixed rate of interest. Note that V is a functional depending on the whole time paths of p and A . The optimal policies must satisfy the initial conditions $p(0) = p_o$ and $A(0) = A_o$.

It is important to note that the optimal policies need not be continuous functions of time. For example, suppose that A can be chosen without any restriction such as (1) and that z is fixed. The initial stock of good will does not, then, constitute an effective constraint. Thus the optimal policies at any time will be made under the same conditions and must therefore be the same, i.e., constant. Since the optimal choice of A may not be A_o , there will be a discontinuity at $t = 0$. The optimal policy will be to increase or decrease A at once to its optimal level and will therefore imply an infinite instantaneous rate of current advertising outlays, $a(0)$. For such paths, the integral in (5) must be interpreted with some care.

Since net revenue depends only on current price, it is clear that if there are no restrictions on price changes, the initial price does not matter. Furthermore, the maximum of V can be found by first maximizing it with respect to price, holding A fixed, and then maximizing the result with respect to A by an appropriate choice of the time path of a . Thus, optimal price policy is determined by maximizing current net revenue.

with respect to price, i.e., by equating marginal gross revenue to marginal production costs <u>at all times</u>.:

$$(6) \qquad (p - C')\frac{\partial f(p,A,z)}{\partial p} + f(p,A,z) = 0 \quad .$$

If we let

$$\eta = -\frac{p}{f}\frac{\partial f}{\partial p}$$

be the elasticity of demand with respect to price, (6) can be written

$$(6') \qquad p = \eta C'/(\eta - 1) \quad ,$$

the usual price formula for a monopolist. If we solve (6) for the optimal price policy $\hat{p}$ as a function of A and z ,

$$\hat{p}(t) = P(A,z) \; ,$$

and insert the result in (5), we obtain a new problem, namely, to maximize

$$(7) \qquad \hat{V}\{A\} = \int_0^\infty e^{-\alpha t}[\hat{R}(A,z) - a]dt \quad ,$$

subject to (1) and the initial condition $A(0) = A_o$. Note that (1) determines a if A is given; hence, an optimal solution for A gives an optimal solution for a .

2. <u>Determination of Optimal Advertising Policy</u>[7]

Since A_o is fixed, maximizing the surplus

$$(8) \qquad S\{A\} = \hat{V}\{A\} - A_o \quad ,$$

subject to (1) and $A(0) = A_o$ is equivalent to the problem stated in the previous section. Expanding (8) by means of (1) and (7), we have

$$(9) \qquad S\{A\} = \int_0^\infty e^{-\alpha t}[\hat{R}(A,z) - \dot{A} - \delta A]dt - A_o$$

$$= \int_0^\infty e^{-\alpha t}[\hat{R}(A,z) - \delta A]dt - [A_o + \int_0^\infty e^{-\alpha t}\dot{A}\,dt] \; .$$

Integrating the second term on the right by parts, we find

$$(10) \qquad A_o + \int_o^\infty e^{-\alpha t} \dot{A}\, dt = A_o + \left[e^{-\alpha t} A(t)\right]_o^\infty + \alpha \int_o^\infty e^{-\alpha t} A\, dt$$

$$= \lim_{t \to \infty} [e^{-\alpha t} A(t)] + \alpha \int_o^\infty e^{-\alpha t} A\, dt .$$

Substituting in (9), we obtain

$$(11) \qquad S\{A\} = \int_o^\infty e^{-\alpha t}[\hat{R}(A,z) - (\alpha + \delta)A]dt - \lim_{t \to \infty} [e^{-\alpha t} A(t)] .$$

The function

$$(12) \qquad \Pi(A,z) = \hat{R}(A,z) - (\alpha + \delta)A$$

may be called the net "profit" function; if good will were ordinary capital it would represent what was left of revenue net of production expenses after deduction of interest and depreciation charges on capital.

We make three assumptions:

<u>Assumption 1</u>: The limit $\lim_{t \to \infty} [e^{-\alpha t} A(t)]$ exists.

<u>Assumption 2</u>: The net profit has a unique local maximum at a value A^* .[8]

<u>Assumption 3</u>: For sufficiently large A , the net profit function is decreasing.

We first assume that z is constant, so that we are considering a <u>stationary environment.</u>

Under Assumptions 1-3, it can be shown that any optimal policy for constant z must be bounded;[9] for consider a policy, A , which is unbounded. By Assumption 3, we can find another policy $\tilde{A}$ which is bounded and for which $S\{A\}$ has a higher value. Let A_M be any value greater than A_o for which the net profit function is decreasing for $A \geq A_M$. Then the policy

$$\tilde{A} = \min\{A_M, A\}$$

is bounded, and by construction,

$$\Pi(\tilde{A},z) \geq (A,z) \tag{13}$$

for all t . Since $\tilde{A}$ is bounded,

$$\lim_{t \to \infty} [e^{-\alpha t}\tilde{A}(t)] = 0 \leq \lim_{t \to \infty} [e^{-\alpha t} A(t)] \ . \tag{14}$$

It follows from (13) and (14) that

$$S\{\tilde{A}\} \geq S\{A\}$$

where A is any unbounded policy. Thus the class of all <u>bounded</u> policies includes the optimal policy and we may restrict our consideration to bounded policies for which $\lim_{t \to \infty} [e^{-\alpha t} A(t)]$ vanishes.

To maximize $S\{A\}$, it is desirable to make $\Pi(A,z)$ as large as possible, subject to the constraints (1) and $A(0) = A_o$. Assumptions 2 and 3 imply that $\Pi(A,z)$ has a unique global maximum which is also the local maximum of Π with respect to A . This occurs at the value A^* which may be determined by solving

$$\begin{aligned} \frac{\partial \Pi(A,z)}{\partial A} &= \frac{\partial \hat{R}(A,z)}{\partial A} - (\alpha + \delta) \\ &= [P(A,z) - C'[f[P(A,z),A,z]] \frac{\partial f[P(A,z),A,z]}{\partial A} \\ &\qquad - (\alpha + \delta) \\ &= 0 \ , \end{aligned} \tag{15}$$

for A where $P(A,z)$ is determined by solving (6) for the optimal price policy $\hat{p}$ as a function of A and z .[10]

We denote the solution to (15) as A^* ; it is a function of z . Since z is, in general, a function of time, A^* has been defined as a function of time. We will refer to the policy

$$A(t) = A^*[z(t)] \tag{16}$$

as being the instantaneously optimal policy. This policy can be given a relatively simple form. Let

$$\beta = \frac{A}{f} \frac{\partial f}{\partial A} \ ,$$

the elasticity of demand with respect to advertising. If C' is expressed in terms of p , from (6'), (15) can be simplified to

$$\frac{A^*}{pq} = \frac{\beta}{\eta(\alpha + \delta)} \ . \tag{17}$$

This is a dynamic counterpart of Dorfman and Steiner's main result [4].

What is the relation between the policy designed to maximize net profits at each instant of time and the optimal policy? First consider the case where $z(t)$ is constant over time so that the instantaneously optimal policy is a constant, $A^* = A^*(z)$. Clearly, the policy is optimal for $t > 0$, since

$$\varepsilon\left\{A^*\right\} = \Pi(A^*,z) \int_{0+}^{\infty} e^{-\alpha t} dt = \frac{\Pi(A^*,z)}{\alpha} \tag{18}$$

is as large as possible and not affected by what happens at the single point $t = 0$, and since, for (17), $\dot{A}^* = 0$ so that the constraint (1) is satisfied:

$$a^* = 0 + \delta A^* > 0 \ , \tag{19}$$

where a^* is the current advertising outlay determined by A^*, for $t > 0$. Thus, the key question is: What happens at $t = 0$? If $A^* \geq A_o$, we either have a jump in A^* at $t = 0$, in which case $\dot{A}(0)$ and therefore $a^*(0)$ are infinite,[11] or $\dot{A}^* = 0$. In either case, the constraints are satisfied. On the other hand, suppose that $A^* < A_o$; then $A(t)$ has a downward jump at $t = 0$ and $\dot{A}(0)$, and therefore $A(0)$ are $-\infty$, contradicting (1). We have proved

<u>Theorem 1</u>: If A^* is a point at which $\pi(A,z)$ reaches a maximum, and if $A^* \geq A_o$, then the optimal advertising policy for constant z is

$$a^* = \delta A^* \qquad \text{for } t > 0 ,$$

and

$$a^* = \begin{Bmatrix} \delta A^* & \text{if } A^* = A_o \\ +\infty & \text{if } A^* > A_o \end{Bmatrix} \text{ for } t = 0 .$$

A^* is determined by equations (6) and (15).

Although good will can have an upward jump at $t = 0$, as we have already observed it cannot have a downward jump in view of the constraint (1). What then is the optimal policy if $A^* < A_o$? Clearly, $\pi(A,z)$ increases as we decrease A as long as A is greater than A^*; hence, the optimal policy must be to decrease A as fast as possible. The greatest rate at which this can be done is given by the equality in (1):

(20) $$\dot{A} + \delta A = 0$$

or

(21) $$A = A_o e^{-\delta t} .$$

At some time, $t = \tau$,

$$A(\tau) = A^* ,$$

namely,

$$\tau = \frac{1}{\delta} \log \frac{A_o}{A^*} \quad . \tag{22}$$

Then the firm will be in the same position looking forward from τ as it would have been had $A^* = A_o$ to begin with; hence, the optimal policy will be to continue with the stationary policy $A(t) = A^*$. We have proved:

Theorem 2: If A^* is a point at which $\Pi(A,z)$ reaches a maximum, and if $A^* < A_o$, then the optimal advertising policy for constant z is

$$a^* = 0 \qquad \text{for} \quad 0 \leq t \leq \frac{1}{\delta} \log \frac{A_o}{A^*}$$

and

$$a^* = \delta A^* \quad \text{for} \quad t > \frac{1}{\delta} \log \frac{A_o}{A^*} \quad ,$$

where A^* is determined by equations (6) and (15).

The assumption of a constant z was used in the proofs of Theorems 1 and 2 only to establish that A^* satisfied (1) and that

$$e^{-\alpha t} A(t)$$

approaches zero as t approaches infinity. Without any change in the argument, we state

Theorem 3: Let $A^*(z)$ maximize $\Pi(A,z)$ with respect to A. If

$$\text{(A)} \quad \lim_{t \to \infty} e^{-\alpha t} A^*[z(t)] = 0 \quad ,$$

and

$$\text{(B)} \quad \dot{A}^*[z(t)] + \delta A^*[z(t)] \geq 0 \quad \text{for all} \quad t \quad ,$$

then the optimal policy, $A(t)$, may be described as follows:

(a) for $A_o < A^*[z(0)]$, the policy consists of a jump at $t = 0$ to $A^*[z(0)]$; $A(t) = A^*[z(t)]$ for $t > 0$;

(b) if $A_o = A^*[z(0)]$, then $A(t) = A^*[z(t)]$ for all $t \geqq 0$;

(c) if $A_o > A^*[z(0)]$, then $A(t) = A_o e^{-\delta t}$ for $0 \leqq t \leqq \tau$, and $A(t) = A^*[z(t)]$ for $t \geqq \tau$, where τ is a solution (if any) of the equation

$$A_o e^{-\delta t} = A^*[z(t)] ;$$

(d) if $A_o > A^*[z(0)]$ and $A_o e^{-\delta t} \geqq A^*[z(t)]$ for all $t \geqq 0$, then $A(t) = A_o e^{-\delta t}$ for all $t \geqq 0$.

In terms of current advertising expenditures, $a^*(0) = +\infty$ in case (a), and $a^*(t) = 0$ for $0 \leqq t \leqq \tau$ in case (c); otherwise, $a^*(t)$ is given by the right-hand side of (B).

If (B) does not hold, the optimal solution becomes complicated. It may become profitable to have $A(t)$ fall below the instantaneously optimal level, even at times when this policy does not violate (1), in order to prepare for later intervals in which (1) is violated. A special case of this problem, with no depreciation and a finite time horizon, has been studied in [2], and even in this case the algorithm cannot be described simply.

3. Some Comparative Dynamics of the General Solution for a Stationary Environment

At the level of generality of the model described in the previous sections, it is possible to discuss the effects of changes in the two parameters, α and δ, on the optimal stationary policies $\hat{p}$, a^* or A^* , and on τ , the time point at which these stationary policies are achieved, when z is assumed constant. To go further, it is necessary to specialize the model and we shall do this in the next section.

First note that α and δ enter symmetrically into equation (15) which determines A^*, and affect $\hat{p}$ only insofar as they affect A^*. Hence, as far as the effects of variations in the two upon either $\hat{p}$ or A^* are concerned, they are alike. Since $\pi(A,z) = \hat{R}(A,z) - (\alpha + \delta)A$ has a unique maximum (both global and local) with respect to A, an increase in either α or δ must decrease the optimal stationary policy A^*. Analytically, the result follows from the fact that, at $A = A^*$, $\partial^2\pi/\partial A^2 = \partial^2 \hat{R}/\partial A^2 \leqq 0$.

The effect of a change in either α or δ upon the optimal price policy follows directly from the fact that an increase in either is equivalent to a decrease in A^*. Unfortunately, the effect of a decrease in A^* upon the optimal price depends upon its effect on the elasticity of demand, as is well-known. Hence, we cannot specify the result without more specific knowledge of the demand function.

Since A_o is fixed, the effect of a change in α or δ upon τ, for $\tau > 0$, may be determined by differentiating (22):

(23a) $$\frac{\partial\tau}{\partial\alpha} = -\frac{1}{\delta}\,\frac{\partial A^*}{\partial\alpha}/A^* > 0 \quad , \text{ for } \tau > 0 ,$$

since $\delta > 0$, $A^* > 0$, and $\frac{\partial A^*}{\partial\alpha} < 0$.

(23b) $$\frac{\partial\tau}{\partial\delta} = -\frac{1}{\delta}\left\{\frac{\partial A^*}{\partial\delta}/A^* + \tau\right\}, \text{ for } \tau > 0 ,$$

which is positive or negative, according as

(24) $$\left|\frac{\partial A^*}{\partial\delta}/A^*\right| \gtrless \tau, \text{ for } \tau > 0 ,$$

since $\frac{\partial A^*}{\partial\delta} < 0$. Thus, an increase in the interest rate must always postpone achievement of the stationary policy A^*, but an increase in the depreciation rate may actually hasten it. The reason is simply that although an increase in the depreciation rate will lower A^* still further below A_o (assuming that it is below so that $\tau \neq 0$), it also permits a faster approach with zero current advertising expenditures. It may be observed that $\partial\tau/\partial\delta > 0$ for τ sufficiently small, i.e., A_o not too far above equilibrium, and negative in the contrary case.

The effects of changes in α and δ upon optimal current advertising expenditures in the case in which $\tau = 0$ (in other words, for the stationary part of the solution) may be found by differentiating (19):

(25) $$\frac{\partial a^*}{\partial \alpha} = \delta \frac{\partial A^*}{\partial \alpha} < 0 \quad , \text{ for } \tau = 0 \, , \, t > 0 \quad ,$$

and

(26) $$\frac{\partial a^*}{\partial \delta} = \delta \frac{\partial A^*}{\partial \delta} + A^* \quad , \text{ for } \tau = 0 \, , \, t > 0 \quad ,$$

which is positive or negative, according as

(27) $$\left| \frac{\delta}{A^*} \frac{\partial A^*}{\partial \delta} \right| \lessgtr 1 \quad ,$$

that is, according as A^* is inelastic or elastic with respect to δ. Again the ambiguity in the effect of a change in δ results from the fact that, although an increase in δ reduces the optimal A^* by increasing the opportunity cost of good will, such an increase also implies a higher level of current advertising expenditures to maintain any given level of good will.

4. Some Comparative Dynamics of the Solution in a Special Case

It is plain from the preceding discussion that not much can be said about the comparative dynamics of the solution to the optimal advertising problem in the general case. If, however, one is willing to set his sights lower and specify particular forms for the demand and cost functions, a great deal more can be deduced. In this section it will be assumed that $\tau = 0$, that the total production cost function is linear in q, and that the demand function is of a particular multiplicative form,

$$f(p,A,z) = f_1(p)\, f_2(A)\, f_3(z) \,. \tag{28}$$

For still more definite results, we will assume that the demand function is linear in the logarithms, which is a special case of (28):

$$f(p,A,z) = kp^{-\eta} A^{\beta} z^{\zeta} \,. \tag{29}$$

Previously, the symbols η and β have been defined as the elasticities of demand with respect to price and good will, respectively; ζ is the elasticity of demand with respect to the variable, z, ; if z is income, then ζ is the income elasticity of demand.

Theorem 3 assures us that the optimal policy for all t will coincide with the instantaneously optimal policy defined by (31) and (33) below, at least after a finite time period, provided A^* neither increases nor decreases too rapidly over time.

Since we are assuming that the total cost function is linear, the marginal cost is constant.

$$C' = \gamma \text{ , a constant.} \tag{30}$$

Under the multiplicative assumption (28), η depends only on the variable p and β on A. Then, when (30) holds, (6') is an equation involving only p (and not A or z), so that

(31) $\hat{p} = \gamma\, \eta/(\eta-1)$ is a constant with respect to A and Z.

Notice that (31) implies that η is also a constant with respect to A and z. Since the optimal price is surely not negative, we must have $\eta > 1$.

Under the assumptions (28) and (30), we can write, from (4),

$$R(p,A,z) = (p-\gamma)\, f_1(p)\, f_2(A)\, f_3(z) \tag{32}$$
$$= R_1(p)\, f_2(A)\, f_3(z) \quad ,$$

where

$$R_1(p) = (p-\gamma)\, f_1(p) \quad .$$

The price, $\hat{p}$, maximized $R_1(p)$; let $\hat{R}_1 = R_1(\hat{p})$. Then $\hat{R}(A,z) = \hat{R}_1\, f_2(A)\, f_3(z)$. If we now apply (15), we have

$$f_2'(A^*) = (\alpha + \delta)/\hat{R}_1\, f_3(z) \quad , \tag{33}$$

and the second-order condition for an optimum is that $f_2''(A^*) < 0$. If we differentiate (33) with respect to α , we have

$$f_2''(A^*)\,(\partial A^*/\partial\alpha) = 1/\hat{R}_1\, f_3(z) \quad ,$$

and therefore

$$\partial A^*/\partial\alpha = \partial A^*/\partial\delta < 0 \quad . \tag{34}$$

For constant z , we can apply (25) to find

$$\partial a^*/\partial\alpha = \delta(\partial A^*/\partial\alpha) < 0 \quad . \tag{35}$$

If, on the other hand, we substitute (34) into (26), the sign remains ambiguous.

For any given p , $R_1(p)$ is a decreasing function of γ ; hence, the maximum value, R_1 , must also decrease as γ increases. Then the right-hand side of (33) increases with γ ; since $f_2'' < 0$,

$$\partial A^*/\partial\gamma < 0 \quad . \tag{36}$$

To find the effect of γ on the price, p , we note that the latter is defined by the condition, Under the assumptions (28) and (30), we can write, from (4),

$$R(p,A,z) = (p-\gamma)\, f_1(p)\, f_2(A)\, f_3(z) \tag{32}$$
$$= R_1(p)\, f_2(A)\, f_3(z) \quad ,$$

where

$$R_1(p) = (p-\gamma)\, f_1(p) \quad .$$

The price, $\hat{p}$, maximized $R_1(p)$; let $\hat{R}_1 = R_1(\hat{p})$. Then $\hat{R}(A,z) = \hat{R}_1\, f_2(A)\, f_3(z)$. If we now apply (15), we have

$$f_2'(A^*) = (\alpha + \delta)/\hat{R}_1\, f_3(z) \quad , \tag{33}$$

and the second-order condition for an optimum is that $f_2''(A^*) < 0$. If we differentiate (33) with respect to α , we have

$$f_2''(A^*)\, (\partial A^*/\partial\alpha) = 1/\hat{R}_1\, f_3(z) \quad ,$$

and therefore

$$\partial A^*/\partial \alpha = \partial A^*/\partial \delta < 0 \quad . \tag{34}$$

For constant z , we can apply (25) to find

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If, on the other hand, we substitute (34) into (26), the sign remains ambiguous.

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$$\partial A^*/\partial \gamma < 0 \quad . \tag{36}$$

To find the effect of γ on the price, p , we note that the latter is defined by the condition,

$$\partial R_1/\partial p = 0 \quad ,$$

and that the second-order condition for a maximum is $\partial^2 R_1/\partial p^2 < 0$.

Then, as usual, we have

$$\frac{\partial^2 R_1}{\partial p^2}\frac{dp}{d\gamma} + \frac{\partial^2 R_1}{\partial p \partial \gamma} = 0 \quad .$$

Since

$$\frac{\partial^2 R_1}{\partial p \partial \gamma} = - f_1'(p) > 0 \quad ,$$

(37) $$dp/d\gamma > 0 \quad .$$

To sum up, it has been established that for multiplicative demand functions and constant marginal costs, price is constant with respect to income, the rate of interest or the rate of depreciation, and increases with marginal cost γ , the stock of good will decreases as the rate of interest or the rate of depreciation or the marginal cost increases, and current advertising outlay decreases as the rate of interest increases for fixed z .

We now move to the still more specific hypothesis, (29), that the demand function is linear in the logarithms. The second-order conditions now imply that $\beta < 1$. From (17), we now find that the optimal good will is a constant proportion of sales. But note that, since z may be changing over time, sales may not remain constant, in which case A^* will also change.

We may solve (29), (31), and (17) for A^* as a function of z :

(38) $$A^* = \left[\frac{k \beta \gamma^{1-\eta}}{(\eta-1)(\alpha+\delta)} \right]^{1/(1-\beta)} \left(\frac{\eta}{\eta-1} \right)^{-\eta/(1-\beta)} z^{\zeta/(1-\beta)} \quad .$$

Thus A^* is a function solely of the time path of income and of the parameters, α , β , γ , δ , η , and ζ . The parameter k may be thought of as defining the units of z ; hence, we may set $k=1$ without loss of generality.

If z is not held constant over time, the results given by (35) no longer hold because the effects of increasing income may offset the effects of changes in the interest rate or depreciation rates. Differentiating (38) with respect to time and substituting the result in (1), we find

$$(39) \qquad a^*(t) = [\delta + \frac{\zeta}{1-\beta}(\frac{dz}{dt}/z)]A^*$$

where A^* is given by (38). We will suppose that income (or whatever other demand shifter z is taken to represent) is changing or expecting to change at a constant proportional rate, ρ , so that

$$(40) \qquad \frac{dz}{dt}/z = \rho \text{ , constant.}$$

Under these conditions, it may easily be verified that the assumptions (A) and (B) of Theorem 3 are equivalent to the conditions,

$$(41) \qquad \alpha > \zeta\rho/(1-\beta) \geqq -\delta \quad ,$$

and these will be assumed in the following. (If the second inequality is reversed, then obviously the optimal policy is never to advertise.)

Thus, we may consider the effects of changes in seven parameters: α , β , γ , δ , η , ρ , and ζ on optimal current advertising expenditure. Rather than discuss all the possible effects, however, we shall limit ourselves to the effects of changes in the interest and depreciation rates, α and δ , and leave the remaining analyses to the reader.

Differentiating a^* with respect to α , we find

$$(42) \qquad \frac{\partial a^*}{\partial \alpha} = [\delta + \frac{\zeta\rho}{1-\beta}] (\frac{(-A^*)}{(\alpha+\delta)(1-\beta)}) = \frac{-a}{(\alpha+\delta)(1-\beta)} < 0 \quad ,$$

so that an increase in the interest rate always reduces optimal current advertising expenditure. On the other hand, differentiating with respect to δ , we find

$$(43) \qquad \frac{\partial a^*}{\partial \delta} = \frac{[(1-\beta)\alpha - \delta\beta] - \frac{\zeta\rho}{1-\beta}}{(\alpha+\delta)(1-\beta)} A^* \quad .$$

Thus, the effect of a change in the depreciation rate upon optimal current advertising expenditures depends upon the relationships among all the parameters.

More useful conclusions may be drawn by expressing optimal advertising expenditures as a ratio to sales. Let this ratio be σ ; substituting for A^* in (39), from (17), we obtain

$$(44) \qquad \sigma = \frac{\beta}{\eta(\alpha+\delta)} (\delta + \frac{\zeta\rho}{1-\beta}) \quad .$$

If the value of σ computed from (44) is negative, then the optional policy is zero advertising. Thus, assuming, as we have, that income (or any other demand shifter) changes at a constant proportional rate (which may be zero) implies that firms should try to keep a constant ratio of sales to advertising. It is interesting to note that firms really do seem to follow this rule of thumb. Borden [3, pp. 721-22] reports that in 1935 the Association of National Advertisers found that of 215 companies investigated, 54 per cent stated that their advertising appropriations were determined as a percentage of sales, either of the past year or expected in the year of the budget, and another 16 per cent stated that their appropriations were guided in part by a percentage of sales. While the evidence is meager, what there is does suggest that the model developed here is plausible.

The parameters of the model enter into the determination of the optimal ratio of advertising to sales in a much simpler manner than they do in the determination of the absolute level of optimal current advertising. It is therefore easier to discuss the effects of changes in these parameters on the optimal ratio. The derivatives of the ratio with respect to these parameters are given in the table below.

Table. Effects of Changes in $\alpha, \beta, \gamma, \delta, \eta, \rho$, and ζ upon σ.

Parameter	Derivative of σ with respect to the parameter	Sign of the derivative of σ
α	$\frac{-\sigma}{\alpha + \delta}$	< 0
β	$\frac{(1-\beta)^2\delta + \eta(\alpha+\delta)\zeta\rho}{\eta(\alpha+\delta)(1-\beta)^2}$	> 0
γ	0	$= 0$
δ	$\frac{\beta}{\eta(\alpha+\delta)^2}(\alpha - \frac{\zeta\rho}{1-\beta})$	> 0
η	$-\frac{\sigma}{\eta}$	< 0
ρ	$\frac{\beta}{\eta(\alpha+\delta)}(\frac{\zeta}{1-\beta})$	> 0
ζ	$\frac{\beta}{\eta(\alpha+\delta)}(\frac{\rho}{1-\beta})$	> 0 if $\rho > 0$ < 0 if $\rho < 0$

Note that changes in marginal cost have no effect, an increase in the interest rate or in the price elasticity of demand always reduces the ratio, and that an increase in advertising effectiveness or in the rate of growth of income always increases the ratio. The effects of a change in the depreciation rate, δ, always increases the proportion, in contrast to its effect on the absolute level which is ambiguous. The effect of a change in the income elasticity of demand depends on the sign of ρ.

5. Summary

In this paper we derive optimal advertising and price policies for the individual firm under conditions of imperfect competition. Our model is simplified in the sense that it allows jump policies, which imply an infinite rate of current advertising expenditure, in the initial period. More realistic models may be developed by setting an upper bound on the rate of expenditure (Arrow [1]) or by introducing a non-linear investment cost function (Strotz [9] and Nerlove [8]). Our model has the advantage of considering contraction as well as expansion policies.

When no factors operate to shift the demand function independently of the firm's actions, we show that the optimal price and advertising policies are stationary after a certain point. The concept of good will, a stock related to the flow of current advertising expenditures, is introduced. We assume that this stock depreciates at a constant proportional rate, δ, and that the future is discounted at a constant rate of interest, α. Although we find that changes in α and δ affect the optimal good will in the same way, the effects upon current advertising expenditures and the time at which a stationary policy commences are asymmetrical.

Analysis of the special case in which demand is linear in logarithms and total cost is linear leads to more specific conclusions. We show that the optimal stationary solution implies a constant ratio of advertising to sales. Even in the non-stationary case, in which other factors operate to shift the demand function, the same result is obtained when these factors are assumed to change at a constant proportional rate. This result is supported by some evidence on the actual behavior of firms.

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Footnotes

1. The idea that advertising is a form of investment occurs in Hoos [5].

2. See Borden [3], pp. 433-38.

3. Vidale and Wolfe [10] present a large amount of empirical evidence that the effects of advertising linger on but diminish as time passes. As Waugh [11] has put it,"...old advertisements never die -- they just fade away." Vidale and Wolfe studied a number of situations in which an advertising campaign was run and then all further advertising ceased; they were thus able to ascertain the extent to which the effect of the campaign diminished over time.

4. The assumption that the cost of adding to good will is always one, no matter at what level current advertising expenditures are carried on, is actually very unrealistic. At very high levels of current advertising expenditure resort must be had to inferior media so that the costs of adding a dollar's worth to good will must surely rise with the level of expenditure. One possible way of dealing with this problem is to set a finite upper bound to current

advertising expenditure, below which we assume a proportional cost of adding to good will. This alternative has been discussed by Arrow [1]. Alternatively and more generally, one might introduce a non-linear cost function for additions to good will. This procedure has been used in Strotz [9] and Nerlove [8] in connection with optimal investment policies. Lack of one or the other of these assumptions leads to policies which may have a jump at $t = 0$. Since we are primarily interested in the characteristics of the optimal policy after $t = 0$, however, we shall restrict ourselves to the simpler, but more unrealistic case.

5. One can conceive of a situation in which the effects of negative advertising expenditure on demand could be achieved, namely, let a firm advertise the product of a competitor. Unfortunately, to achieve the same effect on net revenue as negative advertising, one's competitor would have to pay double the amount of the expenditure to the firm in question. This is hardly plausible. For this reason, and since good will cannot be sold in any other way without selling the entire firm, we rule out negative current advertising expenditures altogether.

6. The concept of exponentially depreciating good will was essentially introduced by Waugh [11]. It leads in the discrete case to a distributed lag model similar to the one employed by Jastram [6] which in turn was based on the work of Koyck [7].

7. This section is largely based on the material given in [1].

8. Arrow [1] deals with the more general situation in which the net profit function may have a finite number of distinct local maxima.

9. Note, however, that the magnitude which is bounded is good will, not current advertising expenditure. It is perfectly possible for the latter to be unbounded even if the former is bounded.

10. Equation (15) is a generalization of the result obtained by Dorfman and Steiner [4]. It states that at the optimal price (price equal marginal production costs), the marginal revenue from increased good will net of the marginal costs of producing the increased output should be equal to the marginal opportunity cost of investment in good will. To see this, note that the instantaneous rate of return on investment is α ; the instantaneous decay is δ ; therefore, if the firm invests a dollar now and spends it on advertising later, it makes α and saves δ . To put the matter another way, a dollar invested in a bond will yield $e^{\alpha t}$ in t periods, whereas a dollar invested in advertising will require $e^{-\delta t}$ further investment over the t periods to offset decay, the opportunity cost is $e^{\alpha t} - e^{-\delta t}$, so that the marginal opportunity cost at $t = 0$ is $\alpha + \delta$.

11. This is the difficulty referred to in footnote 4. Modifying the model in either of the two ways suggested there would lead to a policy without a jump.

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